cutoff M for the upper limit. Then we have

$$\Delta=(1/2\pi)[I_+-I_-], \tag{A4}$$

where

$$I_\pm=\int_0^M dE'\frac{\ln(E'-E_0\pm i\Gamma/2)}{E'-E-i\eta}. \tag{A5}$$

The substitution

$$z\equiv E'-E_0\pm i\Gamma/2$$

permits us to write

$$I_\pm=\int_{C_\pm}\frac{dz\ \ln z}{z-a_\pm}, \tag{A6}$$

where

$$a_\pm=E-E_0\pm i\Gamma/2. \tag{A7}$$

The contours $C_\pm$ are illustrated in Fig. 3.

As $|E-E_0|$ and Γ become very small the points $a_\pm$ approach the branch point. Only I_+ becomes singular in this case. Its singularity may be exhibited by moving the contour up into the positive imaginary z plane and keeping the residue of the pole at a_+. The leading (singular) term in I_+ is

$$I_+\cong 2\pi i\ \ln a_+,$$

or

$$\Delta\cong -\ln a_+, \tag{A8}$$

from which Eq. (4.5) follows.

PHYSICAL REVIEW VOLUME 136, NUMBER 5B 7 DECEMBER 1964

Analysis of Low-Energy K^+-p Elastic Scattering*

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An analysis based on dispersion-relation techniques is applied to experimental data for K^+-p elastic scattering. Particular reference is made to the "force of longest range" due to the exchange of low-mass pion pairs with isospin $I=0$. The effect of this exchange force can be calculated in terms of only one unknown parameter λ which may essentially be chosen to be a linear combination of the K-π scattering lengths. The other forces of shorter range are described by further undetermined parameters. The K^+-p differential cross section is calculated in terms of these parameters and a minimization procedure is used to obtain a fit to the experimental data. A good fit is obtained for a well-defined set of values of the parameters. In particular, λ is well determined. A sum rule for $K\pi$ scattering is used to calculate a further relation between the K-π scattering lengths so that the value of each of these is obtained.

I. INTRODUCTION

THE data for the elastic scattering of K^+ mesons on protons[1,2] indicate that the interaction is dominantly s wave and repulsive up to laboratory momenta of 800 MeV/c.[3] This may be seen from the phenomenological, pure s wave, fits which accurately reproduce the data in this energy region.[1,4] This paper is concerned with an analysis of the experimental data using dispersion relation techniques. Previous analyses[5,6] of the K-N interaction along these lines took explicit account of the $I=1$, $J=1$ ρ-meson exchange force and assumed that this was the dominant long-range contribution. However, since the completion of these calculations of Ferrari *et al.*[5] and Lee,[6] the location of the ρ resonance has been found to be $\sim$750 MeV, rather than the lower value of $\sim$500 MeV that they used, and also the existence of the ω resonance has been established at roughly the same energy as the ρ. The KN exchange force arising from these resonances is thus harder to separate from other "short-range" forces, for example, those associated with hyperon and hyperon-resonance exchange.

* The major part of this work was performed while both of the authors were at the University of Illinois, and the work was supported in part by the U. S. Office of Naval Research.

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Several calculations[7] have been performed with the assumption that the dominant exchange forces in KN scattering arise only from the single "particle" intermediate states in the crossed $K\bar{K}$-$N\bar{N}$ and $\bar{K}N$-$\bar{K}N$ processes, i.e., ρ, ω, Λ, Σ, Y_0^*, Y_1^* exchange forces. Even with such drastic assumptions, it has been impossible to draw any quantitive conclusions, owing to the large number of unknown coupling constants that enter such analyses. An analysis of KN scattering has also been made[8] in which the exchange of two nonresonating pions has been calculated in fourth order perturbation theory.

Previous authors[6,9] have pointed out that in the complex s plane, the singularity due to the $K\bar{K}$-$\pi\pi$-$N\bar{N}$ interaction extends extremely close to the physical cut; thus, the exchange of pion pairs with energies close to the threshold value might be expected to produce by far the strongest energy and angle dependence in the physical KN amplitudes. The observed isotropy of the low-energy K^+p data in both energy and angle should imply a considerable limitation on the magnitude of such long-range exchange forces.

In the present calculation the effect of the short-range forces is parametrized and the long-range contribution from the $K\bar{K}$-$\pi\pi$-$N\bar{N}$ process is calculated explicitly. It will be seen that the contribution arising from nonresonant pion pairs near their threshold mass, in the latter process, may be evaluated in terms of known quantities except for one unknown quantity which is essentially a linear combination of the K-π scattering lengths. By performing a χ^2 fit to all the available K^+-p differential cross-section measurements up to 800 MeV/c, an optimum determination of the unknown parameters was made. A sum rule was used to determine a different linear combination of the K-π scattering lengths and so each of these scattering lengths was obtained. Besides the determination of the K-π scattering lengths, attempts were also made to obtain information about the coupling of the ρ and ω mesons to the $K\bar{K}$ and $N\bar{N}$ systems. Unfortunately, the data is, at present, too meager to determine the extra parameters introduced when the ρ and ω exchange is treated explicitly.

To calculate the differential cross section, the N/D method was used to evaluate the partial-wave amplitudes for s-wave ($J=\frac{1}{2}$) and p-wave ($J=\frac{1}{2}$ and $\frac{3}{2}$) scattering. The discontinuity across the near left-hand cut was determined explicitly; the effect of the remaining singularities was represented by two poles (for each partial amplitude) whose positions were chosen by a method due to Balázs,[10] and whose residues were unknown parameters. For each of the p-wave amplitudes, one of these residues was determined by the threshold behavior.

An estimate of the contributions to the differential cross section from higher partial waves ($l>1$) was made by writing a dispersion relation for K^+-p scattering in $\cos\theta$ and subtracting off the two lowest partial waves. Only the two-pion exchange term was kept in this dispersion relation, but since two subtractions had been made, neglect of the more distant singularities should be reasonable.

The effect of Coulomb scattering was included, and the final parametric form of the K^+-p differential cross section was obtained. The differential cross section was a function of five parameters, four of which were the arbitrary pole residues approximating the contributions of the short-range forces to s and p waves. The remaining parameter appears in the description of the long-range forces arising from the exchange of the low-mass pion pairs. If explicit account was also taken of the ρ and ω exchange by separating them from the other short-range forces, a further two parameters were introduced. An IBM-7094 computer was used to obtain a best fit to the data and so find the optimum values of the parameters.

The available "low-energy" K^+p data consist of 52 differential cross-section measurements at nine different K^+ laboratory momenta in the range 140–810 MeV/c.[1,2] The data at higher energies were not used in the present analysis because the competition of inelastic processes is expected to increase rapidly above 800 MeV/c. It should be noted that the amount of information to be obtained from the analysis of K^+p scattering would be greatly increased by measurements of the polarization of the recoil proton at K^+ incident laboratory momenta up to 750 MeV/c. There are polarization measurements at 910 MeV/c,[11] but because of the value of the incident momentum and the large errors attached to these results, they are of little practical value in this analysis.

In Sec. II we introduce the kinematics of the KN system and discuss the singularities of the partial wave amplitudes in the s and ν planes. In Sec. III, the processes $K\bar{K}\rightarrow\pi\pi$ and $K\pi\rightarrow K\pi$ are discussed. Fixed momentum transfer dispersion relations and a sum rule are obtained for the latter process. The $I=0$, $J=0$ two-pion exchange contribution and the ρ and ω-exchange terms are formulated. Section IV describes the parametrization of the K^+-p differential cross section, and Sec. V contains a discussion of the results obtained when this parametric form is fitted to the K^+p data.

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[9] F. Ferrari, G. Frye, and M. Pusterla, Phys. Rev. **123**, 308 (1961).

[10] L. A. P. Balázs, Phys. Rev. **125**, 2179 (1962).

[11] W. Hirsch and G. Gidal, Phys. Rev. **135**, B191 (1964).

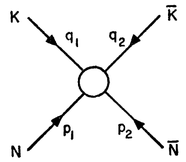

FIG. 1. KN scattering and related processes.

II. THE AMPLITUDES FOR KN SCATTERING

This section is concerned with the kinematics of the KN system. The angular momentum decompositions of the amplitudes describing KN scattering and the crossed processes are carried out and the singularities of the KN partial-wave amplitudes are discussed.

(a) Kinematics

The three channels to be considered in a dispersion relation treatment of KN scattering are obtained by choosing different incident pairs of the particles shown in Fig. 1. q_1, q_2, p_1, and p_2 are the ingoing 4-momenta of the particles shown in Fig. 1. The three reactions are

$$\begin{aligned} &(\mathrm{I}) \quad K+N\to K+N\,,\\ &(\mathrm{II}) \quad \bar{K}+N\to \bar{K}+N\,,\\ &(\mathrm{III}) \quad K+\bar{K}\to N+\bar{N}\,. \end{aligned}$$

Convenient variables for the description of this system are the squares of the total center-of-mass energies in the three channels; these are written s, u, and t, respectively, and are given by[12]

$$s=-(q_1+p_1)^2,\quad u=-(q_1+p_2)^2,\quad t=-(q_1+q_2)^2.$$

It can be seen that $s+u+t=2M^2+2m^2$ and thus only two of these variables are independent. We denote the nucleon, K-meson, and π-meson masses by M, m, and μ, respectively.

The two independent variables may alternatively be chosen to be the magnitude of the three-momentum and the scattering angle in the barycentric system of any one of the processes. For reaction I,

$$\begin{aligned} s&=M^2+m^2+2k^2+2[(M^2+k^2)(m^2+k^2)]^{1/2},\\ t&=-2k^2(1-\cos\theta)\,, \end{aligned} \tag{2.1}$$

where k is the magnitude of the 3-momentum and θ the scattering angle in the barycentric KN frame. For reaction III,

$$\begin{aligned} s&=-p^2-q^2-2pq\cos\theta_3\,,\\ t&=4(p^2+M^2)=4(q^2+m^2)\,, \end{aligned} \tag{2.2}$$

where θ_3 is the scattering angle and p and q are the magnitudes of the 3-momentum of the nucleon and kaon in the center-of-mass frame for reaction III.

For any of the three channels the S matrix may be written in the form

$$S_{fi}=\delta_{fi}-i\frac{(2\pi)^4M\delta^4(p_1+q_1+p_2+q_2)}{(4p_{10}q_{10}p_{20}q_{20})^{1/2}}\tau_{fi}\,, \tag{2.3}$$

[12] We use the metric $pq=\mathbf{p}\cdot\mathbf{q}-p_0q_0$.

where for channel I

$$\tau_{fi}=\bar{u}(-p_2)Tu(p_1)\,, \tag{2.4}$$

and for channel III

$$\tau_{fi}=\bar{u}(-p_2)Tv(-p_1)\,, \tag{2.5}$$

where u and v are positive and negative energy solutions of the Dirac equation normalized to $u\bar{u}=1$ and $v\bar{v}=1$.

The customary decomposition of the matrix T into scalar amplitudes is as follows:

$$T=-A+\tfrac{1}{2}i\gamma\cdot(q_1-q_2)B\,, \tag{2.6}$$

$$\begin{aligned} A&=A^+I+A^-\tau_N\cdot\tau_K\,,\\ B&=B^+I+B^-\tau_N\cdot\tau_K\,. \end{aligned} \tag{2.7}$$

$A^\pm(s,t)$, $B^\pm(s,t)$ are the scalar invariant amplitudes which are assumed to satisfy a Mandelstam representation. They are related to the amplitudes A^I, B^I for isospin $I=0$, 1 in channels I and III by the relations

$$A_{\mathrm{I}}{}^0=A^+-3A^-,\qquad A_{\mathrm{I}}{}^1=A^++A^-, \tag{2.8a}$$

$$A_{\mathrm{III}}{}^0=2A^+,\qquad A_{\mathrm{III}}{}^1=2A^-. \tag{2.8b}$$

For convenience we shall omit the isospin superscripts in the remainder of this section.

(b) Partial-Wave Decomposition

The partial-wave analysis in channel I is completely analogous to that for pion-nucleon scattering.[13] For completeness the relevant results are summarized in Eqs. (2.9) to (2.12). Omitting the modification in the presence of a Coulomb interaction, the barycentric kaon-nucleon differential cross section is given by

$$d\sigma(k,x)/d\Omega=|f_1(k,x)+xf_2(k,x)|^2+(1-x^2)|f_2(k,x)|^2, \tag{2.9}$$

where the variables k and $x\equiv\cos\theta$ have been introduced in Eqs. (2.1). The partial-wave expansions may be written in the form

$$\begin{aligned} F_1&\equiv f_1+xf_2=\sum_{l=0}^{\infty}[(l+1)f_{l+}(\nu)+lf_{l-}(\nu)]P_l(x)\,,\\ F_2&\equiv -f_2=\sum_{l=1}^{\infty}[f_{l+}(\nu)-f_{l-}(\nu)]P_l{}'(x)\,, \end{aligned} \tag{2.10}$$

where $\nu=k^2$. Thus, the partial-wave amplitudes are given by

$$f_{l\pm}(\nu)\equiv\frac{e^{i\delta_{l\pm}}\sin\delta_{l\pm}}{k}=\frac{1}{2}\int_{-1}^{1}dx[f_1(k,x)P_l(x)+f_2(k,x)P_{l\pm1}(x)]. \tag{2.11}$$

$\delta_{l\pm}$ is the K-nucleon phase shift for a state with orbital angular momentum l and total angular momentum

[13] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu. Phys. Rev. **106**, 1337 (1957).

$j=l\pm\frac{1}{2}$. The relation between the differential cross section and the invariant amplitudes of Eqs. (2.6) is obtained from the result

$$\begin{aligned} f_1(k,x)&=\frac{(W+M)^2-m^2}{16\pi W^2}\\ &\qquad\times[A(s,t)+(W-M)B(s,t)],\\ f_2(k,x)&=\frac{(W-M)^2-m^2}{16\pi W^2}\\ &\qquad\times[-A(s,t)+(W+M)B(s,t)], \end{aligned}\tag{2.12}$$

with $W\equiv\sqrt{s}$.

In channel III the angular momentum decomposition is analogous to that for the process $\pi\pi\to N\bar{N}$.[14] In the center-of-mass frame the $K\bar{K}\to N\bar{N}$ differential cross section, for the production of a nucleon with helicity λ and an antinucleon with helicity λ', may be written as

$$d\sigma/d\Omega=|f_{\lambda\lambda'}(\theta_3,\varphi_3)|^2,\tag{2.13}$$

where the helicity amplitudes $f_{\lambda\lambda'}$ are given by[15]

$$\begin{aligned} f_{++}(\theta_3,\varphi_3)&=\frac{1}{q}\sum_J(J+\tfrac{1}{2})T_{++}{}^J(t)P_J(\cos\theta_3),\\ f_{+-}(\theta_3,\varphi_3)&=\frac{1}{q}\sum_J(J+\tfrac{1}{2})T_{+-}{}^J(t)\\ &\qquad\times\frac{e^{-i\varphi_3}\sin\theta_3}{[J(J+1)]^{1/2}}P_J'(\cos\theta_3). \end{aligned}\tag{2.14}$$

Here $T_{\lambda\lambda'}{}^J(t)=-iS_{\lambda\lambda'}{}^J(t)$ and $S_{\lambda\lambda'}{}^J(t)$ is the submatrix of S_{fi} for total angular momentum J and center-of-mass total energy $W_t\equiv\sqrt{t}$. The variables relevant to channel III were introduced in Eq. (2.2).

Alternatively, from Eqs. (2.3) and (2.13) it can be shown, on making a choice of phase, that

$$f_{\lambda\lambda'}=(-M/4\pi W_t)(p/q)^{1/2}\tau_{fi}.\tag{2.15}$$

Substituting Eq. (2.6) into (2.5) and reducing the Dirac spinors to Pauli spinors χ_λ, Eq. (2.15) becomes

$$f_{\lambda\lambda'}=-\frac{M}{4\pi W_t}\left(\frac{p}{q}\right)^{1/2}\chi_{\lambda'}{}^\dagger[h_1\boldsymbol{\sigma}\cdot\mathbf{p}+h_2\boldsymbol{\sigma}\cdot\mathbf{q}]\chi_{-\lambda},$$

where

$$\begin{aligned} h_1&=-\frac{1}{M}\left[A+B\frac{\mathbf{p}\cdot\mathbf{q}}{M+\frac{1}{2}W_t}\right],\\ h_2&=\tfrac{1}{2}(W_t/M)B. \end{aligned}\tag{2.16}$$

For comparison with the partial-wave expansions, Eqs. (2.14), we evaluate Eq. (2.16) for specific values of λ and λ' by inserting the appropriate Pauli spinors. We find,

$$\begin{aligned} f_{++}&=\frac{M}{4\pi W_t}\left(\frac{p}{q}\right)^{1/2}\{h_1p+h_2q\cos\theta_3\},\\ f_{+-}&=\frac{M}{4\pi W_t}\left(\frac{p}{q}\right)^{1/2}h_2q\sin\theta_3e^{-i\varphi_3}. \end{aligned}\tag{2.17}$$

Comparing Eqs. (2.14) and (2.17) the following decomposition of the invariant amplitudes in channel III is found, with $y=\cos\theta_3$,

$$\begin{aligned} A(s(y),t)&=-\frac{4\pi W_t}{p}\sum_J\frac{(J+\frac{1}{2})}{(pq)^{1/2}}\Big\{T_{++}{}^J(t)P_J(y)\\ &\qquad-\frac{2M}{W_t}T_{+-}{}^J(t)\frac{yP_J'(y)}{[J(J+1)]^{1/2}}\Big\},\\ B(s(y),t)&=\frac{8\pi}{q}\sum_J\frac{(J+\frac{1}{2})}{(pq)^{1/2}}T_{+-}{}^J(t)\frac{P_J'(y)}{[J(J+1)]^{1/2}}. \end{aligned}\tag{2.18}$$

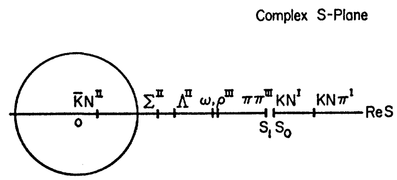

FIG. 2. The singularities of the KN partial-wave amplitudes in the complex s plane.

(c) Singularities of the KN Partial-Wave Amplitudes

The invariant amplitudes $A(s,t)$, $B(s,t)$ are assumed to satisfy a Mandelstam representation and thus the analytic properties of the partial-wave amplitudes $f_{l\pm}$ may be obtained from Eqs. (2.11) and (2.12). The positions of the singularities of the amplitudes $f_{l\pm}$ in the complex s plane corresponding to the intermediate states of the three channels were first derived by MacDowell.[16] These are reproduced in Fig. 2 on which the points labelled in the form P^X refer to the branch points, nearest the physical threshold $s=s_0\equiv(M+m)^2$, arising from intermediate states P in channel X. Thus, the interaction of longest range for KN scattering would be expected to arise from the two-pion intermediate state in reaction III. The branch point nearest to the physical threshold due to the exchange of a system of total energy W_t in channel III is given by

$$s(W_t)=M^2+m^2-t/2+2[(M^2-t/4)(m^2-t/4)]^{1/2},$$

and thus we find $s_1\equiv s(2\mu)=0.96s_0$. Although an intermediate state of definite mass in channel I is associated with one point on the right-hand (physical) cut, the singularities arising from such a state in channels II and III are smeared out, owing to the partial-wave projec-

[14] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960); see also Ref. 9.

[15] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[16] S. W. MacDowell, Phys. Rev. **116**, 774 (1959); see also J. Kennedy and T. D. Spearman, *ibid.* **126**, 1596 (1962).

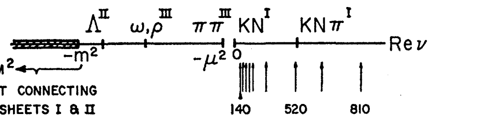

FIG. 3. Singularities in $f_{l\pm}(\nu)$ in the 1st ν sheet. The arrows on the right-hand cut indicate energies at which the K^+-p differential cross section has been measured experimentally (Refs. 1 and 2); the numbers give the corresponding laboratory momenta of the K^+ in MeV/c.

tion, and appear over ranges of the left-hand cut, one section always extending to $-\infty$. In the present calculation, the absorptive part of the amplitude is calculated explicitly only over the portion of the left-hand cut from s_1 to just below the threshold for the ρ and ω contribution. Thus, it is more convenient to work with the variable $\nu=k^2$, as the singularities in $f_{l\pm}(\nu)$ are all on the real axis in the complex ν plane. However, the complex s plane is mapped into a two-sheeted ν plane,[17] connected by the cut $-M^2\leq\nu\leq-m^2$; the region of the s plane, $s(+)$, exterior to the circle cut, maps onto sheet I and the interior region, $s(-)$, maps onto sheet II of the ν plane

$$s(\pm)=[(\nu+M^2)^{1/2}\pm(\nu+m^2)^{1/2}]^2.$$

The singularities of $f_{l\pm}(\nu)$ in the 1st ν plane are shown in Fig. 3. In the calculation to be described below, the absorptive part of the amplitude is calculated explicitly only for values of ν in the range $-m^2<\nu_L\leq\nu\leq-\mu^2$ and an approximation is made to the discontinuity across the remainder of the left-hand cut; consequently a dispersion relation for $f_{l\pm}(\nu)$ can be written in the first ν sheet alone. However, if ν_L were less than $-m^2$ it would then be necessary to also write a dispersion relation in the second ν sheet, because for ν in the range $-M^2<\nu<-m^2$ it is not possible to straightforwardly identify the discontinuity across the cut with the absorptive part of an amplitude.

III. THE DISCONTINUITY ACROSS THE NEARBY LEFT-HAND CUT

In this section the discontinuities in the KN partial-wave amplitudes across the nearby cut, that is $\nu_L\leq\nu\leq-\mu^2$, are related to the amplitudes for the process $\pi\pi\rightarrow N\bar{N}$ and for $K\pi$ scattering. In subsection (a), amplitudes for $K\bar{K}\rightarrow\pi\pi$ are calculated in terms of the K-π scattering amplitudes. In (b), the unitarity relation is used to evaluate the discontinuities in the KN amplitudes in terms of the amplitudes for $K\bar{K}\rightarrow\pi\pi$ and $\pi\pi\rightarrow N\bar{N}$. Finally, in (c), the contributions to the discontinuities arising from ρ and ω exchange are given.

(a) The $K\bar{K}\rightarrow\pi\pi$ Amplitudes

The three related processes[18] $K\pi\rightarrow K\pi$, $K\pi\rightarrow K\pi$, and $K\bar{K}\rightarrow\pi\pi$ are labeled channels I, II, and III, respectively. The kinematics[19] are analogous to those introduced in Sec. II(a) for the KN system. Equations (2.1) and (2.2) are unchanged[20] except that M is replaced by μ and p by κ, where κ is the magnitude of the pion momentum in channel III. For boson-boson scattering the conventional analog of Eq. (2.3) is

$$S_{fi}=\delta_{fi}-i\frac{(2\pi)^4\delta^4(p_1+q_1+p_2+q_2)}{(16p_{10}q_{10}p_{20}q_{20})^{1/2}}T_{fi}. \quad (3.1)$$

The decomposition of T into scalar amplitudes is

$$T=\delta_{\beta\alpha}T^+ +\tfrac{1}{2}[\tau_\beta,\tau_\alpha]T^-, \quad (3.2)$$

where α, β are the isospin indices of the pions. The amplitudes $T^\pm$ are related to the amplitudes T^{2I} for isospin $I=\frac{1}{2}$, $\frac{3}{2}$ in channel I by

$$T^1=T^+ +2T^-,\quad T^3=T^+-T^-, \quad (3.3a)$$

and to the channel III amplitudes T^I for isospin $I=0$, 1 by

$$T^0=(6)^{1/2}T^+,\quad T^1=2T^-. \quad (3.3b)$$

The partial-wave decomposition in channel I takes the form

$$T^{2I}(s,t)=-8\pi(s)^{1/2}\sum_l(2l+1)P_l(\cos\theta)f_l^{2I}(s), \quad (3.4)$$

where $f_l^{2I}(s)\equiv(e^{i\delta}\sin\delta)/k$, with

$$\delta\equiv\delta_{2I,2l}. \quad (3.5)$$

In channel III the decomposition is written

$$T^\pm(s,t)=\sum_l(2l+1)(\kappa q)^l g_l^\pm(t)P_l(\cos\theta_3), \quad (3.6)$$

where the partial-wave amplitudes are thus given by

$$g_l^\pm(t)=\tfrac{1}{2}(\kappa q)^{-l}\int_{-1}^{1}d(\cos\theta_3)T^\pm(s,t)P_l(\cos\theta_3). \quad (3.7)$$

The amplitudes $g_l(t)$ are analogous to the amplitudes $f_\pm^J(t)$ introduced by Frazer and Fulco[14] for the process $\pi\pi\rightarrow N\bar{N}$, and in particular the factor $(\kappa q)^l$ has similarly been introduced to remove the threshold zero. The summation in Eq. (3.6) runs over even (odd) values of l for the $+(-)$ amplitude due to G-parity conservation.

In order to investigate the exchange of $I=0$, $l=0$ pion pairs in KN scattering, the amplitude $g_0^+(t)$ must be determined for use in the unitarity relation [see Eq. (3.16)]. The partial-wave amplitude $g_0^+(t)$ is an analytic function in the t plane except for the right-hand branch cut $4\mu^2\leq t<\infty$ related to channel III, and the left-hand cut $-\infty<t\leq0$, arising from channels I and

[17] R. Oehme, Phys. Rev. Letters 4, 246 (1960), and Erratum: 4, 320 (1960); J. Hamilton and T. D. Spearman, Ann. Phys. (N. Y.) 12, 172 (1961).

[18] One of us (A.D.M.) wishes to thank Dr. L. L. J. Vick for useful discussions concerning the $K\pi$ system.

[19] See, for example, B. W. Lee, Phys. Rev. 120, 325 (1960); M. Gourdin, Y. Noirot, and Ph. Salin, Nuovo Cimento 18, 651 (1960).

[20] Notation: In Sec. III(a), the variables s, t, k, θ, θ_3 are to be interpreted as defined in the $K\pi$ system; in the remainder of the paper, these variables refer to the KN system. Throughout the paper, the variables $\mathbf{q}$, $\mathbf{p}$, $\boldsymbol{\kappa}$ refer to the K, N, π 3-momenta, respectively, in channel III (in either the KN or $K\pi$ systems).

II, i.e., $K\pi$ scattering. The application of unitarity in channel III shows that the phase of $g_0^+(t)$, for $4\mu^2 \leq t < 16\mu^2$, is the phase shift for $I=l=0$ $\pi\pi$ scattering. Thus, the product function Dg_0^+ will not possess the cut $4\mu^2 \leq t < 16\mu^2$ where $D(\nu_\pi)$ is the $I=l=0$ $\pi\pi$ denominator function.[21] Making the "elastic" approximation, that is, assuming the above phase relation to be valid over the entire right-hand cut, a once-subtracted dispersion relation for Dg_0^+ gives the following expression for g_0^+:

$$g_0^+(t)=\frac{1}{D(\nu_\pi)}\left[D(\nu_\pi=-\mu^2)g_0^+(0)+\frac{t}{\pi}\int_{-\infty}^{0}dt'\frac{D(\nu_\pi')\,\mathrm{Im}g_0^+(t')}{t'(t'-t)}\right], \quad (3.8)$$

where $\nu_\pi=t/4-\mu^2=\kappa^2$. Using Eqs. (3.3), (3.4), and (3.7), it can be shown that

$$\mathrm{Im}g_0^+(t)=4\pi\sum_l\frac{2l+1}{\kappa_-q_-}\int_{s_0}^{L(t)}P_l(\cos\theta')(\sqrt{s'})\times[\tfrac{1}{3}\,\mathrm{Im}f_l^1(s')+\tfrac{2}{3}\,\mathrm{Im}f_l^3(s')]ds', \quad (3.9)$$

where the limits are $s_0\equiv(m+\mu)^2$ and $L(t)=m^2+\mu^2+2\kappa_-q_--t/2$; and where $\kappa_-=(\mu^2-t/4)^{1/2}$ and $q_-=(m^2-t/4)^{1/2}$, the positive root being taken for $t<4\mu^2$ and $t<4m^2$, respectively. A few comments should perhaps be made about the derivation of Eq. (3.9). Equation (2.2), with p replaced by κ, was used to change the variable of integration to s'. An extra minus sign appears from the fact that $\mathrm{Im}f_l(s)$ means $\mathrm{Im}f_l(s+i\epsilon)$, and $\mathrm{Im}g_0(t)$ means $\mathrm{Im}g_0(t+i\epsilon)$; and a careful examination of Eq. (2.2) shows that, for real $\cos\theta_3$ and $-1\leq\cos\theta_3\leq 1$, $s+i\epsilon$ maps onto $t-i\epsilon$. There is also a factor 2 because both channels I and II are contributing. The channel II contribution is equal to and of the same sign as that from channel I. In the present calculation, the summation in Eq. (3.9) is approximated by retaining only s-wave terms and the contribution due to K^* (the 888 MeV $I=\frac{1}{2}$, $l=1$ $K\pi$ resonance), and by using the following expressions for these:

$$\mathrm{Im}f_0^{2I}(s)=(ks_0/s)(a^{2I})^2, \quad (3.10)$$

$$\mathrm{Im}f_1^1(s)=(2\pi\Gamma_R(s)^{1/2}/k)\delta(s-s_R), \quad (3.11)$$

where a^{2I} is the s-wave $K\pi$ scattering length for the state of isospin I; $(s_R)^{1/2}$ and Γ_R are the K^* resonance energy and half-width at half maximum, respectively. The use of Eq. (3.10) is based on the assumption that a^1, a^3 are small ($\sim\frac{1}{10}\hbar/\mu c$). The best fits to the K^+-p experimental data discussed in Sec. V do give small values of a^1 and a^3. Equation (3.11) follows from a Breit-Wigner formula for a narrow resonance.

The remaining term of Eq. (3.8) to be related to the $K\pi$ parameters, $g_0^+(0)$, may be written, using Eq. (3.7),

$$g_0^+(0)=\frac{1}{4m\mu}\int_{(m-\mu)^2}^{(m+\mu)^2}T^+(s',0)ds'. \quad (3.12)$$

The amplitude $T^+(s',0)$, which corresponds to forward scattering in channel I is evaluated in the unphysical region by using a once-subtracted fixed t dispersion relation

$$T^+(s,0)=T^+(s_0,0)+\frac{1}{\pi}\int_{s_0}^{\infty}ds'\,\mathrm{Im}T^+(s',0)\left\{\frac{1}{s'-s}+\frac{1}{s'+s-\Sigma}-\frac{1}{s'-s_0}-\frac{1}{s'+s_0-\Sigma}\right\}, \quad (3.13)$$

where $\Sigma=2m^2+2\mu^2$. From Eq. (3.4) the subtraction constant is given by

$$T^+(s_0,0)=-8\pi(m+\mu)[\tfrac{1}{3}a^1+\tfrac{2}{3}a^3]. \quad (3.14)$$

On the right-hand side of Eq. (3.13), $\mathrm{Im}T^+(s',0)$ is evaluated using Eq. (3.4), retaining as before only the s-wave terms and the K^* contribution. Inserting the above evaluation of Eqs. (3.9) and (3.12) into Eq. (3.8), the amplitude $g_0^+(t)$ is parameterized in terms of two parameters, the s-wave K-π scattering lengths a^1, a^3. The K^* resonance parameters are regarded as known.

A relation between these two parameters can be obtained by evaluating the unsubtracted fixed t dispersion relation for $T^-(s,0)$ at $s=s_0$,[22]

$$T^-(s_0,0)\equiv-8\pi(m+\mu)[\tfrac{1}{3}a^1-\tfrac{1}{3}a^3]=\frac{m\mu}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'k'^2}\mathrm{Im}T^-(s',0). \quad (3.15)$$

The convergence of the unsubtracted relation for T^- is comparable to the once-subtracted relation for T^+, Eq. (3.13), because T^- is odd under crossing. As usual only the s-wave terms and K^* contribution are included on the right-hand side of Eq. (3.15). Eliminating one of the scattering lengths, using Eq. (3.15), the $K\bar{K}\to\pi\pi$ partial-wave amplitude, $g_0^+(t)$, is evaluated as a function of one parameter λ, defined to be the remaining s-wave $K\pi$ scattering length. The value of this parameter is estimated from the fit to the K^+-p experimental data described in Sec. V.

The effect of the inclusion of a possible resonance K' (725-MeV $K\pi$ resonance)[23] on the above analysis is also discussed in Sec. V.

(b) The Unitarity Relation for the Process $K\bar{K}\to N\bar{N}$

The unitarity condition is used to determine the discontinuity in the KN partial-wave amplitudes across

[21] $D(\nu_\pi)$ is given by Eq. (26) of T. D. Spearman, Phys. Rev. **129**, 1847 (1963). The $\pi\pi$ pole parameters were taken to be $\Gamma=16$ and $\nu_1=-30$ in the present calculation.

[22] This is the exact analog of the πN sum rule, discussed previously by one of the authors: T. D. Spearman, Nucl. Phys. **16**, 402 (1960).

[23] D. H. Miller, G. Alexander, O. I. Dahl, L. Jacobs, G. R. Kalbfleisch, and G. A. Smith, Phys. Letters **5**, 279 (1963).

the cut $\nu_L \leq \nu \leq -\mu^2$. Since this cut is due to two-pion intermediate states whose mass is below the $K\bar{K}$ and $N\bar{N}$ thresholds, this involves the application of unitarity in an unphysical energy region. It is assumed that the unitarity condition is valid in this region.[24] For two-pion intermediate states, the $K\bar{K} \to N\bar{N}$ unitarity relation is as follows:

$$\frac{1}{2i}[A(s(y),t+i\epsilon)-A(s(y),t-i\epsilon)] = \frac{1}{p^2 W_t}\sum_J (J+\tfrac{1}{2})(pq^*)^J \kappa^{2J+1}\left\{f_+{}^J(t)P_J(y) - \frac{M}{[J(J+1)]^{1/2}}yP_J'(y)f_-{}^J(t)\right\}g_J{}^*(t),$$

$$\frac{1}{2i}[B(s(y),t+i\epsilon)-B(s(y),t-i\epsilon)] = \frac{-1}{pqW_t}\sum_J (J+\tfrac{1}{2})(pq^*)^J\kappa^{2J+1} \times f_-{}^J(t)g_J{}^*(t)\frac{P_J'(y)}{[J(J+1)]^{1/2}}, \tag{3.16}$$

where $f_\pm{}^J(t)$ are the helicity amplitudes for $\pi\pi \to N\bar{N}$ introduced by Frazer and Fulco.[14] The superscript denoting isotopic spin in channel III has been omitted from the amplitudes $A, B, f_\pm{}^J$, and g_J. Equations (2.8b), (3.3b) and the equation for $\pi\pi \to N\bar{N}$ analogous to (3.3b) may be used to write the unitarity relation in terms of the amplitudes $A^\pm, B^\pm, f_\pm{}^{J(\pm)}$, and $g_J{}^\pm$. With the isospin superscript $+(-)$ a factor 3(2) appears on the right-hand side of Eq. (3.16). The derivation of Eqs. (3.16) is discussed in Appendix A. In order to determine the contribution to KN scattering from the exchange of an s-wave pion pair, only the $J=0$ term is retained and the discontinuity becomes

$$\begin{aligned} \mathrm{Im}A_{\pi\pi}{}^+(s,t+i\epsilon) &= (3\kappa/2p^2W_t)f_+{}^0(t)g_0{}^*(t), \\ \mathrm{Im}B_{\pi\pi}{}^+(s,t+i\epsilon) &= 0. \end{aligned} \tag{3.17}$$

As $f_+{}^0$, g_0, and D^* all have the same phase for $4\mu^2 \leq t < 16\mu^2$, it can be shown, for t in this range, that

$$f_+{}^0(t)g_0{}^*(t) = -\mathrm{Im}f_+{}^0(t)[\mathrm{Im}D(\nu_\pi)]^{-1}D(\nu_\pi)g_0(t), \tag{3.18}$$

where $\nu_\pi = \kappa^2 = t/4 - \mu^2$ and $D(\nu_\pi)$ is the $\pi\pi$ denominator function introduced above; in terms of the parameters[21] of the $\pi\pi$ "one-pole" approximation

$$\mathrm{Im}D(\nu_\pi) = -2\Gamma\kappa/W_t(\nu_\pi - \nu_1), \quad \text{for} \quad \nu_\pi \geq 0. \tag{3.19}$$

The discontinuities in the KN partial-wave amplitudes, for $\nu_L \leq \nu \leq -\mu^2$, are obtained by inserting the results of Eqs. (3.17) to (3.19) into Eqs. (2.11) and (2.12).

$$\mathrm{Im}f_{l\pm}{}^1(\nu+i\epsilon) = \frac{+3}{256\pi\nu s\Gamma}\int_{4\mu^2}^{-4\nu} dt \frac{\nu_\pi - \nu_1}{p^2}\mathrm{Im}f_+{}^0(t)D(\nu_\pi)g_0(t) \times\{[(W+M)^2-m^2]P_l(1+t/2\nu) - [(W-M)^2-m^2]P_{l\pm1}(1+t/2\nu)\}. \tag{3.20}$$

As $\nu+i\epsilon$ maps into $t-i\epsilon'$, for ν on the 1st sheet, a minus sign is included in Eq. (3.20). The isospin superscript on the left-hand side refers to the $I=1$ state in channel I, i.e., the state relevant to K^+p scattering. The values for $\mathrm{Im}f_+{}^0(t)$ have been obtained from an analysis of s-wave pion-nucleon scattering.[25] The product Dg_0 is given by Eq. (3.8) and is evaluated in terms of the properties of the $K\pi$ system using the subsequent equations in Sec. III(a). The integrand of Eq. (3.20) can be evaluated, using the approximations outlined above, only for t in the range $4\mu^2 \leq t \leq t_M$. The upper limit is chosen such that at the corresponding $K\bar{K} \to N\bar{N}$ center-of-mass energy, $(t_M)^{1/2}$, the terms arising from the higher partial waves in Eqs. (3.16) are still expected to give a negligible contribution, and also the "elastic" approximation used in Eq. (3.18) is still expected to be a valid approximation. However, contributions to KN scattering from the exchange of systems other than the pion pair will occur for too large a value of t_M; for instance, as the ρ- and ω-exchange contributions have been omitted from the above scheme, t_M should not be greater than $28\mu^2$. Thus Eq. (3.20) is used to determine the discontinuities in KN partial-wave amplitudes (as a function of the $K\pi$ parameter λ) across the left-hand cut, for ν in the range $\nu_L \leq \nu \leq -\mu^2$, where $\nu_L = -\frac{1}{4}t_M$. The further parameters necessary to explicitly include the ρ- and ω-exchange contributions to the discontinuities in the KN amplitudes are briefly discussed in the following subsection.

(c) The Contribution from ρ and ω Exchange

In the approximation where both the ρ- and ω-resonance energies are equal to $(t_R)^{1/2}$, it may be shown that their contribution to the absorptive parts of the invariant amplitudes is

$$\begin{aligned} \mathrm{Im}A_{\rho\omega}{}^1 &= [g^{(2)}/2M](s+\tfrac{1}{2}t-m^2-M^2)\delta(t-t_R), \\ \mathrm{Im}B_{\rho\omega}{}^1 &= -(g^{(1)}+g^{(2)})\delta(t-t_R), \end{aligned} \tag{3.21}$$

where the superscript 1 on the left-hand side refers to

[24] For field-theoretical arguments in favor of this assumption see S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

[25] T. D. Spearman (unpublished); see also J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. 128, 1881 (1962). For the purpose of evaluating the integral in Eq. (3.20), these results are well represented by replacing $\mathrm{Im}f_+{}^0(t)$ by a constant, $\mathrm{Im}f_+{}^0(t) \simeq 24.0$, in units of $\hbar=\mu=c=1$, for $4\mu^2 \leq t \leq 20\mu^2$.

isotopic spin, $I=1$, in channel I, and where[26]

$$g^{(i)}=8\pi^2\{g_{\rho K\bar{K}}g_{\rho N\bar{N}}{}^{(i)}+g_{\omega K\bar{K}}g_{\omega N\bar{N}}{}^{(i)}\},\quad i=1,2. \tag{3.22}$$

In this "pole" approximation, the discontinuity in the amplitudes $f_{l\pm}{}^1(\nu)$ is obtained by substituting Eqs. (3.21) in Eqs. (2.12) and (2.11). Two parameters $g^{(1)}$ and $g^{(2)}$ are thus necessary if the contributions of ρ and ω exchange are to be explicitly taken into account and not included in the approximation to the short-range forces described in Sec. IV.

IV. THE PARAMETRIC FORM FOR THE K^+-p DIFFERENTIAL CROSS SECTION

Several parameters are necessary for our description of low-energy K^+p scattering. One of these, λ, is a K-π scattering length and was introduced in Sec. III. The others are residues of poles which provide a phenomenological description of the short-range forces in K^+p scattering. (There may also be two unknown coupling constants if we treat the ρ and ω explicitly.) The differential cross section $d\sigma/d\Omega(k,x)$ is evaluated in terms of these parameters which are then varied until a fit to the low-energy K^+-p scattering data is obtained. The calculation of $d\sigma/d\Omega(k,x)$ in parametric form is described below; this involves a separate treatment of each of the low partial waves ($j=0$, 1^+ and 1^-) and an approximation to the effect of all the remaining partial waves by a closed term. That is, the amplitudes F_i of Eq. (2.10) are written in the form[27]

$$F_i(k,x)=\bar{F}_i(k,x)+\Delta F_i(k,x),\quad i=1,2, \tag{4.1}$$

where $\bar{F}_i$ is the contribution of the $l=0$ and $l=1$ terms and ΔF_i is the contribution of the remaining partial waves in Eq. (2.10).

(a) Contribution of the Lower Partial Waves

To determine the $\bar{F}_i(k,x)$ defined above, we evaluate the partial-wave amplitudes $f_0(\nu)$, $f_{1+}(\nu)$, and $f_{1-}(\nu)$ individually using the N/D method. Rather than deal with the amplitudes $f_{l\pm}(\nu)$ directly, we define

$$a_{l\pm}(\nu)\equiv(\nu+\mathfrak{M}^2)^{1/2}f_{l\pm}{}^I(\nu)=N_{l\pm}{}^I(\nu)/D_{l\pm}{}^I(\nu), \tag{4.2}$$

where $N(\nu)$ has a cut for $-\infty<\nu\leq-\mu^2$ only, and $D(\nu)$ for $0\leq\nu<\infty$. $\mathfrak{M}$ is an otherwise arbitrary mass satisfying $-\mathfrak{M}^2<\nu_L$. For K^+p scattering the isotopic spin I is 1. The angular momentum and isospin subscripts and superscripts will be omitted.

[26] The coupling constants $g_{\rho K\bar{K}}$ etc., are defined through the interaction Lagrangian

$$\begin{aligned}\mathcal{L}=i(4\pi)^{1/2}[g_{\rho N\bar{N}}{}^{(1)}(\bar{\psi}\gamma_\mu\tau_\lambda\psi)\rho_\mu{}^\lambda+g_{\omega N\bar{N}}{}^{(1)}(\bar{\psi}\gamma_\mu\psi)\omega_\mu]\\+\frac{(4\pi)^{1/2}}{2M}\left[g_{\rho N\bar{N}}{}^{(2)}\bar{\psi}\sigma_{\mu\nu}\tau_\lambda\psi\frac{\partial\rho_\nu{}^\lambda}{\partial x_\mu}+g_{\omega N\bar{N}}{}^{(2)}\bar{\psi}\sigma_{\mu\nu}\psi\frac{\partial\omega_\nu{}^\lambda}{\partial x_\mu}\right]\\+i(4\pi)^{1/2}\left[g_{\rho K\bar{K}}\left(\frac{\partial\phi_K{}^\dagger}{\partial x_\mu}\tau_i\phi_K-\phi_K{}^\dagger\tau_i\frac{\partial\phi_K}{\partial x_\mu}\right)\rho_\mu{}^{(i)}\right.\\\left.+g_{\omega K\bar{K}}\left(\frac{\partial\phi_K{}^\dagger}{\partial x_\mu}\phi_K-\phi_K{}^\dagger\frac{\partial\phi_K}{\partial x_\mu}\right)\omega_\mu\right].\end{aligned}$$

[27] See G. L. Kane and T. D. Spearman, Phys. Rev. Letters **11**, 45 (1963), where a similar decomposition is used for π^+-p scattering.

N and D satisfy the following equations:

$$N(\nu)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2}d\nu'\frac{\operatorname{Im}a(\nu')D(\nu')}{\nu'-\nu}, \tag{4.3}$$

$$D(\nu)=1+\frac{\nu}{\pi^2}\int_{-\infty}^{-\mu^2}d\nu'\operatorname{Im}a(\nu')D(\nu')K(\nu,\nu'), \tag{4.4}$$

where the kernel is found to be,

$$\begin{aligned}K(\nu,\nu')&\equiv\int_0^\infty\left(\frac{\nu''}{\nu''+\mathfrak{M}^2}\right)^{1/2}\frac{d\nu''}{\nu''(\nu''-\nu)(\nu''-\nu')}\\&=\frac{1}{\nu'-\nu}[h(y')/\nu'-h(y)/\nu],\end{aligned} \tag{4.5}$$

with $y=\nu/\nu+\mathfrak{M}^2$ and,

$$\begin{aligned}h(y)&=y^{1/2}\ln\left|\frac{1-y^{1/2}}{1+y^{1/2}}\right|,&&\text{for}\quad y>0,\\h(y)&=2(-y)^{1/2}\tan^{-1}(1/(-y)^{1/2}),&&\text{for}\quad y<0.\end{aligned} \tag{4.6}$$

$\operatorname{Im}f_{l\pm}{}^1(\nu+i\epsilon)$ was calculated in Sec. III for $\nu_L\leq\nu\leq-\mu^2$ under the assumption that the dominant contribution comes from the $I=0$, $J=0$ two-pion intermediate state. The result is given in Eq. (3.20), and $\operatorname{Im}a_{l\pm}{}^1(\nu)$ for ν in the same range is immediately obtained from this and Eq. (4.2). Equations (4.3) and (4.4) are solved for N and D using a procedure proposed by Balazs.[10] In this method the contribution to the integrals corresponding to the range of ν' for which $\operatorname{Im}a(\nu')$ is not determined is approximated by a sum of terms involving unknown parameters α_r, and Eqs. (4.3) and (4.4) become

$$N(\nu)=\frac{1}{\pi}\int_{\nu_L}^{-\mu^2}d\nu'\frac{\operatorname{Im}a(\nu')D(\nu')}{\nu'-\nu}+\sum_r\frac{\alpha_r}{1-\nu/\nu_r}, \tag{4.7}$$

$$D(\nu)=1+\frac{\nu}{\pi^2}\int_{\nu_L}^{-\mu^2}d\nu'\operatorname{Im}a(\nu')D(\nu')K(\nu,\nu')+\frac{\nu}{\pi}\sum_r\alpha_r\nu_rK(\nu,\nu_r). \tag{4.8}$$

These equations are discussed in Appendix B, particular attention being devoted to choosing the optimum values for the ν_r. The Eqs. (4.7) and (4.8) may be solved by the usual techniques and the KN phase shift determined as a function of the parameters α_r and λ, using the equation

$$\delta(\nu)=\tan^{-1}\left[\left(\frac{\nu}{\nu+\mathfrak{M}^2}\right)^{1/2}\frac{N(\nu)}{\operatorname{Re}D(\nu)}\right], \tag{4.9}$$

where $\operatorname{Re}D(\nu)$ denotes the real part of $D(\nu)$. This procedure is carried out for the $j=0$, $1+$, $1-$ partial-wave amplitudes, using in each case two parameters $\alpha_1{}^j$ and $\alpha_2{}^j$ to approximate the unknown "short-range" forces. However, in the case of each of the p-wave

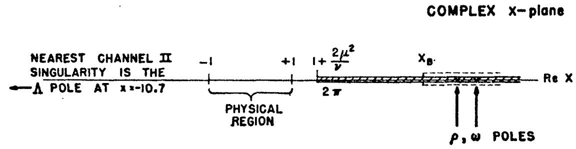

FIG. 4. The singularities of the invariant KN amplitudes in the complex $\cos\theta$ plane for a fixed kaon laboratory momentum of 415 MeV/c. x_B denotes the branch point due to the double spectral functions which is closest to the physical region.

amplitudes, one of the parameters is determined by requiring that $\delta_p(\nu)\sim\nu^{3/2}$ at threshold. Thus, using Eqs. (2.10), the contributions $\bar{F}_i$ to the amplitudes of Eq. (4.1) are obtained in terms of five parameters: α_1^0, α_2^0, α_1^{1+}, α_1^{1-}, and λ.

(b) Contribution of the Higher Partial Waves

The contribution ΔF_i may be expressed in terms of the total amplitudes by simply subtracting off the $l=0$ and $l=1$ terms in the partial-wave expansions of Eq. (2.10); thus,

$$\Delta F_1(k,x)=F_1(k,x)-\frac{1}{2}\int_{-1}^{1}dx'(1+3xx')F_1(k,x'),$$
$$\Delta F_2(k,x)=F_2(k,x)-\frac{3}{4}\int_{-1}^{1}dx'(1-x'^2)F_2(k,x'). \tag{4.10}$$

The singularities of the invariant amplitudes of Eq. (2.7) [and thus, also of the amplitudes $F_i(k,x)$] in the $x\equiv\cos\theta$ plane, at a fixed value of k, are shown in Fig. 4. These singularities move in towards the physical region as k increases, but maintain approximately the form shown in Fig. 4. Since, effectively, two subtractions have been made in the amplitudes F_i, it is reasonable to neglect the distant singularities in the $\cos\theta$ plane and to assume that the following dispersion relation is a good approximation:

$$\Delta F_i(k,x)\simeq\int_{1+2\mu^2/k^2}^{x_M}dx'\frac{\mathrm{Im}\Delta F_i(k,x')}{x'-x}, \tag{4.11}$$

where the range of integration includes only the two-pion cut (i.e., the cutoff is chosen to be in the region $x_M=1+8\mu^2/\nu$). The integrand may be evaluated from Eq. (3.20) by making use of Eqs. (4.10), (2.10), and (2.12); and thus ΔF_i, the contribution from the sum total of the $l\geq 2$ partial waves, is obtained in terms of the parameter λ.

The contributions from the low partial waves $\bar{F}_i$ and the remaining partial waves ΔF_i are combined by means of Eq. (4.1) and the result inserted into Eq. (2.9). Coulomb scattering terms are also included. Thus, the differential cross section is obtained for all K^+ momenta in the low-energy region as a function of the parameters λ, α_1^0, α_2^0, α_1^{1+}, and α_1^{1-}.

V. RESULTS

The parameters introduced in the preceding sections, λ, α_1^0, α_2^0, α_1^{1+}, and α_1^{1-}, are determined by fitting the calculated differential cross section to all the available "low-energy" data for K^+p scattering.[1,2] The fit was achieved by computing the minimum value of the function

$$\chi^2=\sum_i\left[\frac{(d\sigma/d\Omega)_i^E-(d\sigma/d\Omega)_i^C}{E_i}\right]^2, \tag{5.1}$$

with respect to the variations of the parameters. In Eq. (5.1) $(d\sigma/d\Omega)_i^E\pm E_i$ represents the ith experimental result and $(d\sigma/d\Omega)_i^C$ is the corresponding calculated differential cross section which depends on the values of the parameters λ, α_1^0, α_2^0, α_1^{1+}, and α_1^{1-}.[28] The summation in i is over all the experimental data in the "low-energy" region.

It was pointed out in Sec. III that Eq. (3.15) could be used to eliminate one of the s-wave $K\pi$ scattering lengths and that λ would then be taken to be the remaining one. In our calculation, a_1 was eliminated and so λ was chosen to be a_3. It is clear, however, that the linear combination $a^+\equiv\frac{1}{3}a^1+\frac{2}{3}a^3$ occurs naturally in Eq. (3.14): But for the fact that quadratic terms in a^1 and a^3 appear when Eq. (3.10) is inserted into Eq. (3.9), a^+ would be the only combination of a^1 and a^3 that need be involved in the calculation of K^+p scattering. If a^1 and a^3 are small ($\sim$0.1 in units $\hbar/\mu c$), these quadratic terms are relatively unimportant, and so the fit to the data may essentially be regarded as a determination of a^+. Similarly, under such conditions, the sum rule given by Eq. (3.15) essentially determines the linear combination $a^-\equiv\frac{1}{3}a^1-\frac{1}{3}a^3$. The validity of the calculation in Sec. III was seen to depend on the assumption that a^1 and a^3 were small. The solution actually obtained

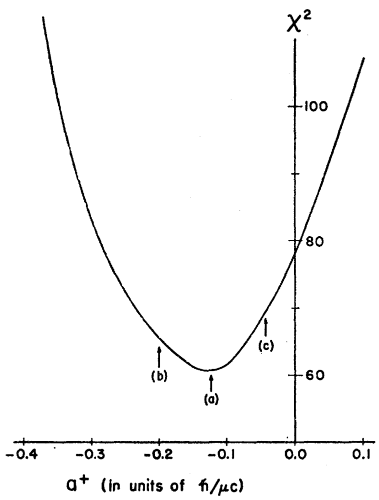

FIG. 5. Plot of χ^2, minimized with respect to α_1^0, α_2^0, α_1^{1-}, α_1^{1+}, against a^+.

[28] The data were found to be insufficient to determine the additional parameters $g^{(1)}$ and $g^{(2)}$ of Eq. (3.21), and thus the ρ and ω contributions were included in the short-range pole parameters.

in this calculation, which will be discussed below, gave the results $a_1=-0.07$, $a_3=-0.16$, and so, is quite consistent with this assumption. For these values, the separation between the fit to the data and the sum rule, as essentially providing independent determinations of a^+ and a^-, respectively, is a meaningful one.

To show how the fit to the data depends on the value of a^+, the minimization procedure described above was performed with respect to $\alpha_1{}^0$, $\alpha_2{}^0$, $\alpha_1{}^{1-}$, $\alpha_1{}^{1+}$ alone, keeping λ, and thus a^+, at various fixed values. The results are shown in Fig. 5. These results were obtained from a fit to the 52 low-energy K^+-p differential cross-section measurements using the following values for the constants appearing in Secs. III and IV: $\nu_L=-7\mu^2$, $\mathfrak{M}^2=7.5\mu^2$, and for the short-range pole positions, $\nu_1=-140\mu^2$, $\nu_2=-10\mu^2$. It should be emphasized that the form and position of the minimum in χ^2 are essentially unaltered if other physically acceptable values are given to these constants.

The s- and p-wave K^+-p phase shifts corresponding to the best fit to the data at fixed values of a^+ are plotted against the center-of-mass momentum in Fig. 6 for the three values of a^+ denoted by (a), (b), and (c) in Fig. 5. The results shown in Fig. 6 confirm the conclusion of Goldhaber *et al.*[1] that the low-energy K^+p interaction

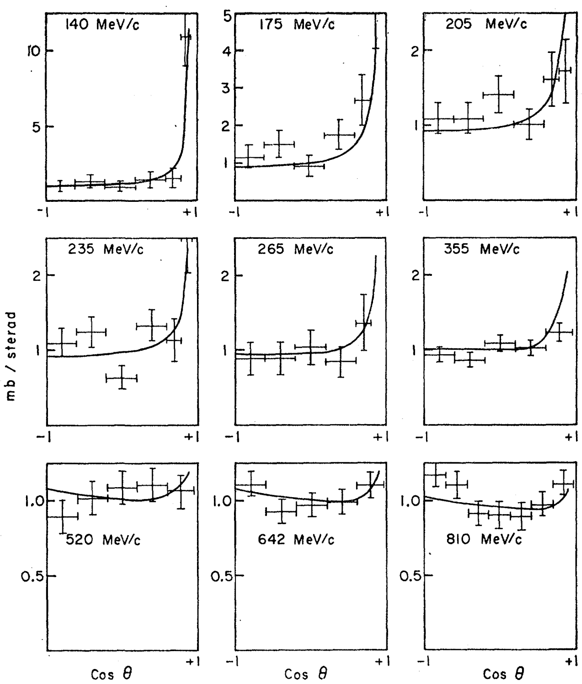

FIG. 7. The best fit to the data ($a^+=-0.125$): $d\sigma/d\Omega$, in mb/sr, versus $\cos\theta$.

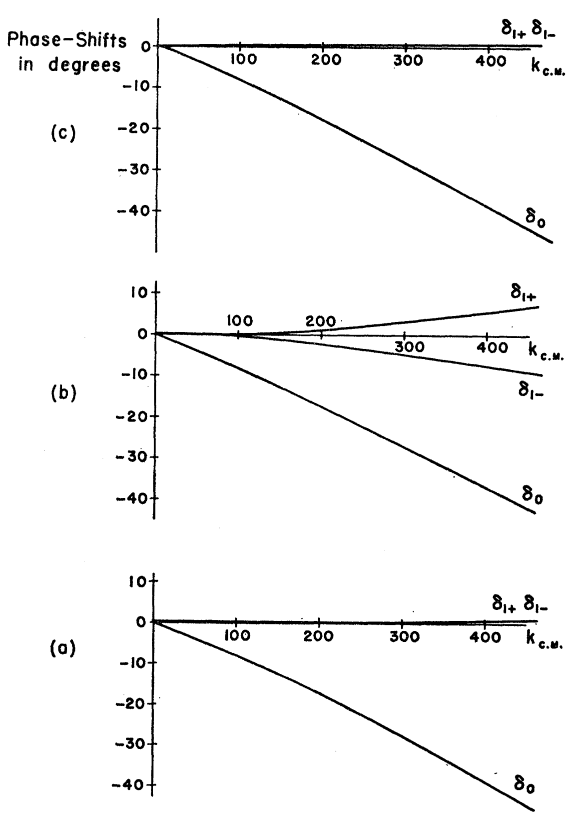

FIG. 6. s- and p-wave $I=1$, KN phase shifts, for values of a^+ denoted by (a), (b), and (c) in Fig. 5 and optimum values of the other four parameters, plotted against the center-of-mass K momentum, $k_{\text{c.m.}}$.

is predominantly an s-wave repulsion. However, it can be seen from the graphs that the p-wave phase shifts can change rapidly with small variations of a^+ near the optimum position so that, with the presently available data, a precise evaluation of these cannot be made. It is probable that they should satisfy the inequalities $|p_{1/2}|<10°$, $|p_{3/2}|<10°$. A precise determination of these p-wave phase shifts would probably require accurate data for the polarization of the recoil proton in K^+p scattering somewhere in the range 500 to 800 MeV/c (lab momentum). Figure 7 shows the optimum fit to the experimental values of the K^+p cross section.

From Fig. 5, regarding the fit to the data essentially as a determination of a^+, the following result is obtained:

$$a^+\equiv\tfrac{1}{3}a^1+\tfrac{2}{3}a^3=-0.13\pm0.03 \text{ in units of } \hbar/\mu c. \quad (5.2)$$

The sum rule, Eq. (3.15), gives for a^-,

$$a^-\equiv\tfrac{1}{3}a^1-\tfrac{1}{3}a^3=+0.03\pm0.05 \text{ in units of } \hbar/\mu c. \quad (5.3)$$

The error quoted in Eq. (5.3) is a probably conservative estimate based on the possible effect of the terms that were omitted from Eq. (3.4) when this was inserted in Eq. (3.15). Equations (5.2) and (5.3) give

$$\begin{aligned} a^1&=-0.07\pm0.10,\\ a^3&=-0.16\pm0.06 \text{ in units of } \hbar/\mu c. \end{aligned} \quad (5.4)$$

Thus, the fit to the K^+p data combined with the sum rule [Eq. (3.15)] indicates a repulsive s-wave $K\pi$ interaction at low energies in both isotopic spin states. It should be noted, however, that for the isospin $I=\frac{1}{2}$

state, the estimated error is sufficiently large that a^1 could have a positive value; and, in particular, the possibility of an s-wave $K\pi$ resonance at 725 MeV cannot be completely ruled out. An effective range formula can give an s-wave resonance at 725 MeV with half-width 5 MeV for a scattering length $a^1 \approx 0.07$.

To see what effect the existence of such a low-energy $K\pi$ resonance in the $I=\frac{1}{2}$ state[23] would have on the present analysis, the fit to the data was repeated, explicitly including the contribution of this resonance (denoted by K'), first, assuming that K' is an s-wave $K\pi$ resonance, and second, assuming that it is in a p-wave state. The K' contribution is included using an expression similar to Eq. (3.11) and taking a resonant half-width of 5 MeV. With the first assumption, the scattering length approximation for $\mathrm{Im} f_0{}^1$ is replaced by the resonance expression; under the second assumption, the K' contribution is added to the K^* contribution of Eq. (3.11).

When an s-wave K' resonance was included, the fit to the data essentially led to the result $a^+ = -0.11$ and the sum rule gave $a^- = 0.04$, with the same errors as before. These values correspond to $a^1 = -0.03$, $a^3 = -0.15$ and are scarcely changed from the results in Eqs. (5.2), (5.3), (5.4) with no K'. As was already mentioned, the possible error in a^1 means that the above value is not inconsistent with the positive value (~ 0.07) which would be expected if there actually were an $I=\frac{1}{2}$, s-wave resonance at 725 MeV with half-width 5 MeV.

With the assumption of a p-wave, $I=\frac{1}{2}$, K' resonance, again with half-width 5 MeV, the best fit to the K^+p data occurred for $a^+ = -0.25$. The sum rule gave $a^- = 0.05$. This corresponds to $a^1 = -0.14$, $a^3 = -0.30$.

It should be pointed out that in the whole of this analysis it was assumed that the K-π scattering lengths were small ($< 0.5\hbar/\mu c$, say). It was shown that a good and unique fit to all the "low-energy" K^+p data could be obtained under this assumption for values of a^1, a^3 given in Eq. (5.4). The possibility of another solution with large values of a^1, a^3 ($\sim 1\hbar/\mu c$) cannot, however, be ruled out in the framework of the present analysis.[29] It should also be stressed that the calculation of the discontinuity across the nearby cut in the K^+-p partial-wave amplitudes, which is of central importance in the present analysis, depends on specific input information about the $I=0$, s-wave π-π phase shift. This input consists of $\mathrm{Im}D(\nu_\pi)$ given by Eq. (3.19) and also, indirectly, $\mathrm{Im}f_+{}^0(t)$ which, although determined from an analysis of low-energy πN scattering, depends also on the form assumed for the $I=0$, s-wave π-π phase shift. The assumption made about the $I=0$, $J=0$ π-π scattering was that this could effectively be approximated by a solution of the N/D equations in which N was represented by a single pole. In Ref. 21 it was shown that the low-energy π-N scattering data and also the so-called ABC anomaly[30] were consistent with this assumption, and the two parameters Γ, ν_1 of the pole were determined from a fit to these data. In the event that the $I=0$, s-wave phase shift has a more complicated behavior,[31] the results described in this section would probably need to be modified.

Finally it should be emphasized that further accurate data on the "low-energy" elastic K^+p scattering could provide more information about both the $K\pi$ and the KN interaction. Accurate measurements of the polarization of the recoil proton for incident K^+ mesons of laboratory momenta in the range 500–800 MeV/c would be particularly useful. The analysis of such data would not only act as a probe to the dynamics of the $K\pi$ interaction, but also should determine the behavior of the K^+-p p-wave amplitudes below 800 MeV/c. This latter information would be of value as it would help us to understand the transition from the s-dominant behavior at momenta below 800 MeV/c to that at higher momenta where p and higher waves and inelasticity become important.

ACKNOWLEDGMENTS

We wish to acknowledge helpful discussions with Professor J. D. Jackson, Dr. G. L. Kane, Professor U. E. Kruse, Professor R. L. Schult, and Dr. L. L. J. Vick. We are grateful to J. T. Donohue for reading the manuscript and checking the equations.

One of us (A.D.M.) would like to acknowledge the hospitality of the Rutherford Laboratory, Berkshire, England, where part of this work was carried out.

APPENDIX A: THE UNITARITY RELATION FOR $K\bar{K} \to N\bar{N}$

The following assumptions are made about unitarity and its relation to the discontinuities across cuts:

(1) That the matrix $T(P_\mu)$, defined by $S(P_\mu) = I(P_\mu) + iT(P_\mu)$, where $S(P_\mu)$ is the submatrix of S corresponding to a total four-momentum P_μ, may be written as

$$T(P_\mu) = \sum_i T_i(P_\mu) O_i, \tag{A1}$$

where the 0_i are Hermitian operators in spin (and isospin) space, such that the matrix elements $T_i(P_\mu)$, between appropriately normalized states, are invariant scalar amplitudes which satisfy a Mandelstam representation. These invariant amplitudes are denoted by $T_i(s,t,u)$, where

$$T_i(s,t,u)\langle \alpha_2 | 0_i | \alpha_1 \rangle = \langle \theta_2\phi_2\alpha_2 | T_i(P_\mu) 0_i | \theta_1\phi_1\alpha_1 \rangle. \tag{A2}$$

[29] Although, if this were the case, it would seem likely that a large enhancement near the $K\pi$ threshold should have been seen in production experiments, e.g., Ref. 23.

[30] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters 7, 35 (1961).

[31] For example, the existence of an $I=J=0$ π-π resonance at 400 MeV has been proposed by some authors: N. P. Samios, A. Bachman, R. Lea, T. Kalogeropoulos, and W. Shephard, Phys. Rev. Letters **9**, 139 (1962); C. Richardson *et al.*, *International Conference on High Energy Physics, CERN, 1962* (CERN Scientific Information Service, Geneva, 1962), p. 96; J. Kirz, J. Schwartz, and R. Tripp, Phys. Rev. **130**, 2481 (1963); L. M. Brown and P. Singer, Phys. Rev. **133**, B812 (1964).

$|\theta_j\phi_j\alpha_j\rangle$ are the appropriate two-particle states, the direction $(\theta_j\phi_j)$ is that of the relative momentum in the center-of-mass system, and α_j are the internal degrees of freedom.

(2) $T_i(s,t,u)_\pm$ denote the boundary values of $T_i(s,t,u)$ at a cut on the real axis, approached from above (+) or below (−), then

$$T_i(s,t,u)_- = T_i^*(s,t,u)_+. \tag{A3}$$

Choosing the states $|\theta_j\phi_j\alpha_j\rangle$, $j=1, 2$, so that

$$\langle\theta_2\phi_2\alpha_2|T_i(P_\mu)0_i|\theta_1\phi_1\alpha_1\rangle = T_i(s,t,u)_+\langle\alpha_2|0_i|\alpha_1\rangle, \tag{A4}$$

the assumption is that

$$\langle\theta_2\phi_2\alpha_2|T_i^\dagger(P_\mu)0_i|\theta_1\phi_1\alpha_1\rangle = T_i(s,t,u)_-\langle\alpha_2|0_i|\alpha_i\rangle. \tag{A5}$$

(3) Finally, it is assumed that unitarity holds in the form

$$SS^\dagger = 1 \tag{A6}$$

whenever there are nonvanishing intermediate states.[32]

For the process $K\bar{K}\to N\bar{N}$ the states $|\theta_j\phi_j\alpha_j\rangle$ are introduced through the equation [cf. Eq. (2.3)]

$$\langle -p_1 - p_2\alpha_N|T|q_1q_2\alpha_K\rangle = \frac{-(2\pi)^4\delta^4(p_1+q_1+p_2+q_2)}{(4p_{10}q_{10}p_{20}q_{20})^{1/2}} \times\langle\theta_2\phi_2\alpha_N|T(P_\mu)|\theta_1\phi_1\alpha_K\rangle, \tag{A7}$$

where the states $|q_1q_2\alpha_K\rangle$ are normalized by

$$\langle q_1'q_2'\alpha_i|q_1q_2\alpha_j\rangle = (2\pi)^6\delta^3(q_1'-q_1)\delta^3(q_2'-q_2)\delta_{ij}. \tag{A8}$$

For this system the 0_i are $-I^sI^I$, $-I^s\boldsymbol{\tau}_N\cdot\boldsymbol{\tau}_K$, $\frac{1}{2}i\gamma(q_1-q_2)I^I$, and $\frac{1}{2}i\gamma(q_1-q_2)\boldsymbol{\tau}_N\cdot\boldsymbol{\tau}_K$, where I^s and I^I are the identity operators in spin and isospin space, respectively. The corresponding amplitudes $T_i(s,t,u)$ are A^+, A^-, B^+, and B^-, respectively.

The partial-wave decomposition is written

$$\langle\theta_2\phi_2\alpha_N|T(P_\mu)|\theta_1\phi_1\alpha_K\rangle = \sum_{J,M}\langle\theta_2\phi_2\alpha_N|JM\alpha_N\rangle \times\langle JM\alpha_N|T(P_\mu)|JM\alpha_K\rangle\langle JM\alpha_K|\theta_1\phi_1\alpha_K\rangle. \tag{A9}$$

With the appropriate choice of α_N and α_K, this equation reduces to Eqs. (2.18). The important observation is that if $T(P_\mu)$ is replaced by $T^\dagger(P_\mu)$ in Eq. (A9), and if this equation is reduced to the form of Eqs. (2.18) using Eqs. (A3) and (A5), then the result is to reproduce Eqs. (2.18), except that A, B on the left-hand side are replaced by A^*, B^*, and $T_{\lambda\lambda'}{}^J(t)$ becomes $T_{\lambda\lambda'}{}^J(t)^\dagger$, where $T_{\lambda\lambda'}{}^J(t)^\dagger$ denotes $\langle JM\alpha_N|T^\dagger(P_\mu)|JM\alpha_K\rangle$. The values of s, t are taken to refer to a point close to the cut with $\mathrm{Im}t>0$, thus,

$$\begin{aligned} A(s,t+i\epsilon)-A(s,t-i\epsilon) &= \frac{-4\pi W_t}{p}\sum_J\frac{(J+\frac{1}{2})}{(pq)^{1/2}}\Big[(T_{++}{}^J(t)-T_{++}{}^J(t)^\dagger)P_J(y) \\ &\quad -\frac{2M}{W_t}(T_{+-}{}^J(t)-T_{+-}{}^J(t)^\dagger)\frac{yP_J'(y)}{[J(J+1)]^{1/2}}\Big], \\ B(s,t+i\epsilon)-B(s,t-i\epsilon) &= \frac{8\pi}{q}\sum_J\frac{(J+\frac{1}{2})}{(pq)^{1/2}} \times(T_{+-}{}^J(t)-T_{+-}{}^J(t)^\dagger)\frac{P_J'(y)}{[J(J+1)]^{1/2}}. \end{aligned} \tag{A10}$$

With the convention $T^J=-iS^J$, the helicity amplitudes introduced in Sec. III for the processes $K\bar{K}\to\pi\pi$ and $\pi\pi\to N\bar{N}$ can be shown to be, respectively,

$$g_J(t) = [-4\pi W_t/(q\kappa)^{1/2}](q\kappa)^{-J}T^J(t), \tag{A11}$$

$$\left.\begin{aligned} f_+{}^J(t) &= \tfrac{1}{2}(p/\kappa)^{1/2}W_t(p\kappa)^{-J}T_{++}{}^J(t)_N, \\ f_-{}^J(t) &= (p/\kappa)^{1/2}(p\kappa)^{-J}T_{+-}{}^J(t)_N. \end{aligned}\right\} \tag{A12}$$

Inserting a complete set of two-pion states, the unitarity relation (A6) becomes

$$T_{\lambda\lambda'}{}^J(t) - T_{\lambda\lambda'}{}^J(t)^\dagger = iT_{\lambda\lambda'}{}^J(t)_N T^J(t)^\dagger. \tag{A13}$$

Applying the same arguments as above and using Eqs. (A11) and (3.7), it is clear that

$$T^J(t)^\dagger = -\frac{(q\kappa)^{1/2}}{8\pi W_t}\int_{-1}^{1} d(\cos\theta_3)T(s,t)^*P_J(\cos\theta_3), \tag{A14}$$

and thus

$$T^J(t)^\dagger = [-(q\kappa)^{1/2}/(4\pi W_t)](q^*\kappa)^J g_J^*(t), \tag{A15}$$

since we are above the two-pion threshold and κ is real. Equations (3.16) now follow from Eqs. (A10), (A12), (A13), and (A15).

APPENDIX B: THE APPROXIMATION TO THE N AND D EQUATIONS

The substitution $y=\nu_L/\nu'$ is made over that part of the range of integration in Eqs. (4.3) and (4.4) for which $\mathrm{Im}a(\nu')$ is unknown, (i.e., $\nu<\nu_L$), and these equations become

$$\begin{aligned} N(\nu) &= \frac{1}{\pi}\int_{\nu_L}^{-\mu^2} d\nu'\frac{\mathrm{Im}a(\nu')D(\nu')}{\nu'-\nu} -\frac{1}{\pi}\int_0^1\frac{dy}{y}\mathrm{Im}a(\nu_L/y)D(\nu_L/y)\frac{1}{1-\nu y/\nu_L}, \\ D(\nu) &= 1+\frac{\nu}{\pi^2}\int_{\nu_L}^{-\mu^2} d\nu'\,\mathrm{Im}a(\nu')D(\nu')K(\nu,\nu') -\frac{\nu\nu_L}{\pi^2}\int_0^1\frac{dy}{y^2}\mathrm{Im}a(\nu_L/y)D(\nu_L/y)K(\nu,\nu_L/y). \end{aligned} \tag{B1}$$

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[32] For a discussion of these assumptions see, for example, D. I. Olive, Nuovo Cimento 26, 73 (1962).

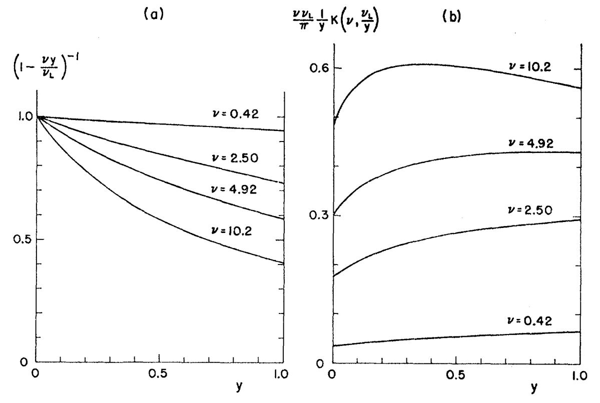

FIG. 8. Plots of $(1-\nu y/\nu_L)^{-1}$ and $\nu\nu_L/\pi\cdot(1/y)K(\nu,\nu_L/y)$ against y over the range $0\leq y\leq 1.0$ for values of ν corresponding to K^+ laboratory momenta of 140, 355, 520, and 810 MeV/c and for $\nu_L=-7\mu^2$, $\mathfrak{M}^2=7.5\mu^2$.

Using an approximation first applied by Balazs,[10] the known part of the kernels of the y integrations are written

$$\frac{1}{1-\nu y/\nu_L}\simeq\sum_{r=1}^{R}\frac{F_r(y)}{1-\nu y_r/\nu_L},$$
$$\frac{1}{y}K(\nu,\nu_L/y)\simeq\sum_{r=1}^{R}\frac{F_r(y)}{y}K(\nu,\nu_L/y_r),\tag{B2}$$

where the values of y_r are chosen, and the functions $F_r(y)$ are such as to make the fit achieve the desired accuracy over the energy range of interest with the minimum value of R. Applying this approximation, the Eqs. (B1) may be written in the form given by Eqs. (4.7) and (4.8) where $\nu_r=\nu_L/y_r$ and

$$\alpha_r=-\frac{1}{\pi}\int_0^1\frac{dy}{y}F_r(y)\ \mathrm{Im}a(\nu_L/y)D(\nu_L/y).\tag{B3}$$

Consider the case where $R=2$, which is the approximation made for the unknown short-range forces in Sec. IV. The values of y_1 and y_2 are to be selected so that the approximation of Eqs. (B2) is as good as possible over the low-energy region for KN scattering, i.e., for ν in the range $0<\nu<10$. To see how this may be accomplished, the kernels to be approximated are plotted over the appropriate range of y for fixed values of ν, a typical plot being shown in Fig. 8. One particular way of making the approximation of Eqs. (B2), with $R=2$, is to choose $F_r(y)$ such that the curves of Fig. 8 are fitted by straight lines, i.e., $F_r(y)=\prod_{s\neq r}(y-y_s)/(y_r-y_s)$. This procedure was adopted in our calculation and y_1, y_2 were given the values $y_1=0.05$ and $y_2=0.7$.

As known functions are approximated in this approach, a certain degree of arbitrariness has been removed from the usual "pole approximation" for the unknown short-range forces; no assumption about the form of $\mathrm{Im}a(\nu)$ on the distant left-hand cut is necessary, although of course some knowledge of its dominant features could be used to improve the accuracy of the approximation.[33]

Finally, a simplification to the method of solution of the N and D equations is mentioned.[34] By substituting into Eqs. (4.8) the expansions

$$N=n_0(\nu)+\sum_r\alpha_r n_r(\nu),$$
$$D=d_0(\nu)+\sum_r\alpha_r d_r(\nu),$$

integral equations are obtained for $n_r(\nu)$ and $d_r(\nu)$, $r=0, 1\cdots R$, that do not contain the parameters α_r. Solving these equations at the start of the variational procedure thus considerably reduces the amount of computation that is necessary.

[33] For instance, if ρ exchange were known to be dominant then it would be desirable to ensure an optimum fit in the region about $y=-4\nu_L/m_\rho^2$. Alternatively one can choose $\mathfrak{M}^2\simeq\frac{1}{4}m_\rho^2$ and so damp out the amplitude in this region. In this calculation $\mathfrak{M}^2$ was given the value $7.5\mu^2$ and so this damping out of the ρ (and ω) was achieved.

[34] H. P. Noyes, Phys. Rev. 119, 1736 (1960).